Internal Gravity Waves (IGW) in the Ocean

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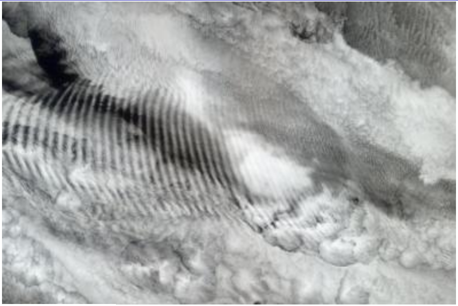
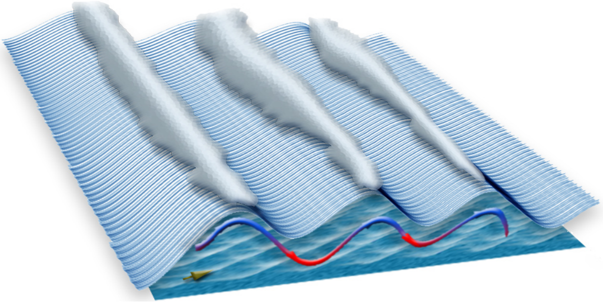
OUTLINE

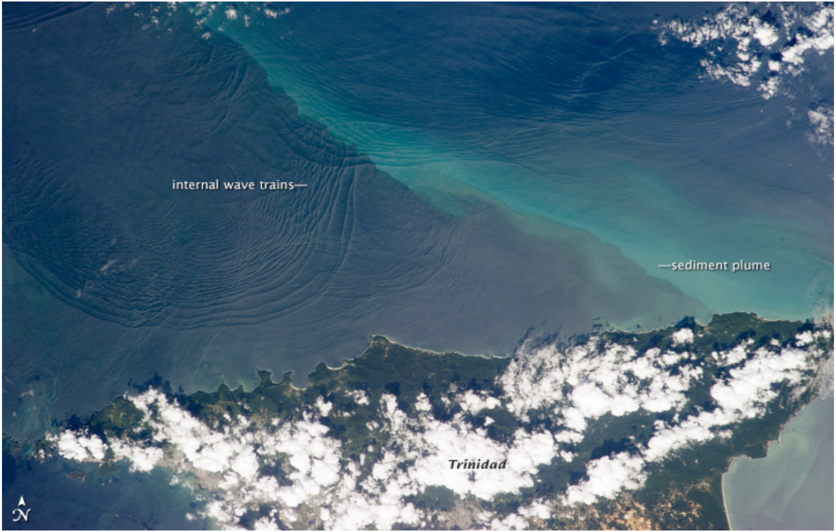
- Introduction
- Mathematical Background
- Surface Gravity Waves
  In-class exercise –> 3 minutes!
- Interfacial Waves
- Internal Waves (Continuous Stratification)
  (One of these days...)

Introduction: Internal gravity waves are everywhere

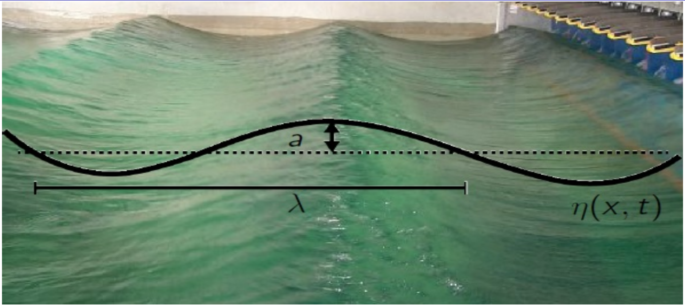




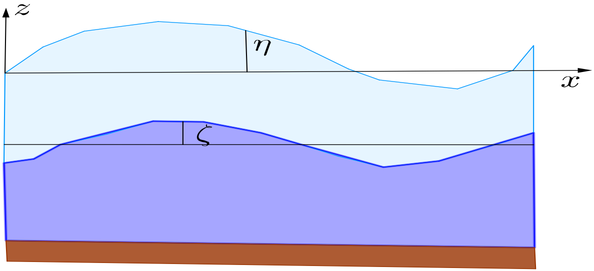
Introduction: Internal gravity waves are everywhere



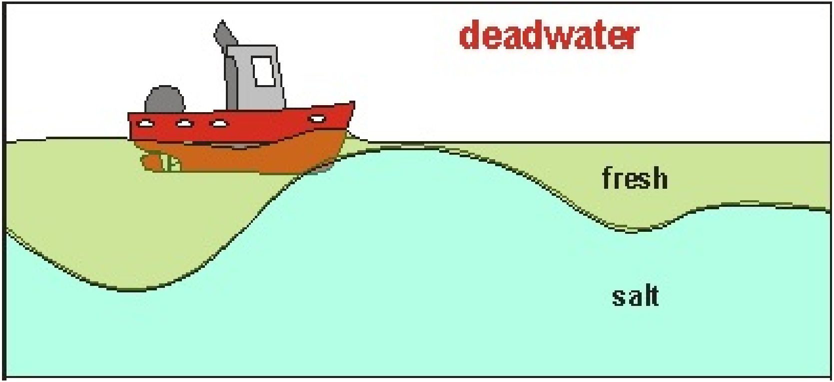
Introduction: Types of IGW



Surface Gravity Waves

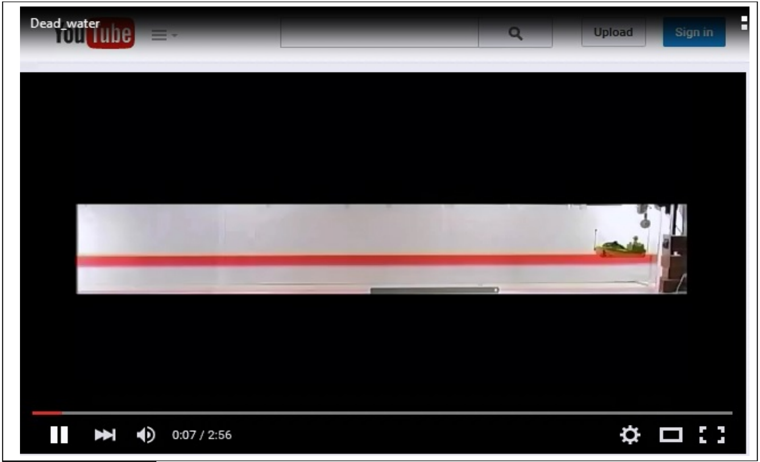


Interfacial Waves



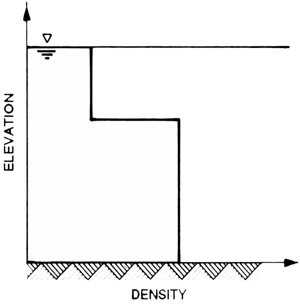
- Ships entering Norwegian fjords experienced increased drag
- It was a mystery for several years and attributed to 'dead water'
- Bjerknes later explained this as due to interfacial waves

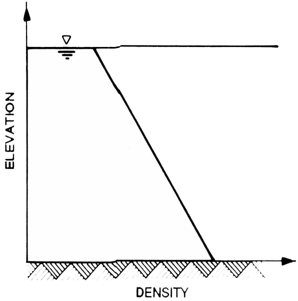
Introduction: Dead water movie



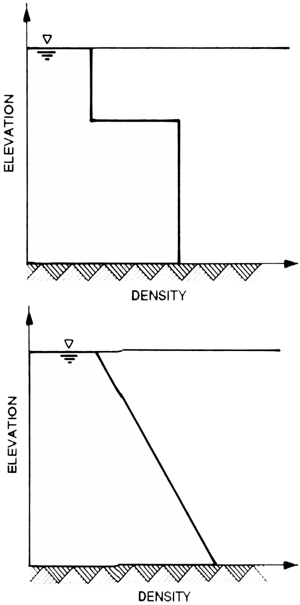
Play/Pause

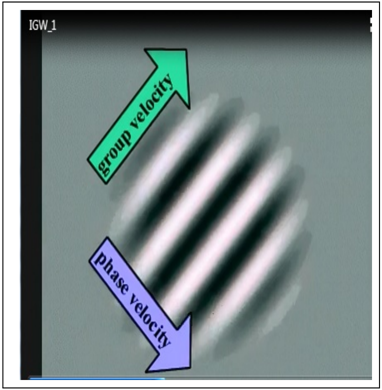
Introduction: Continuously stratified fluid





Introduction: Continuously stratified fluid





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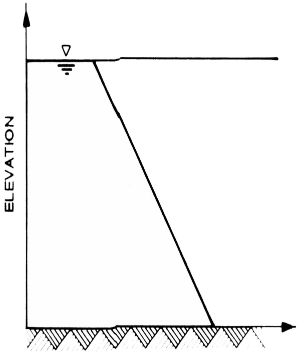
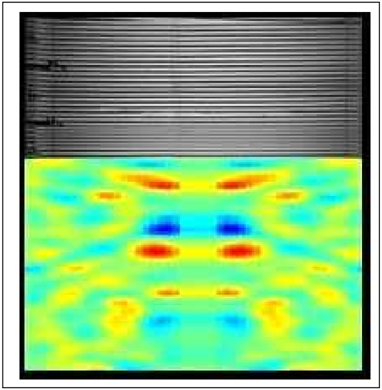
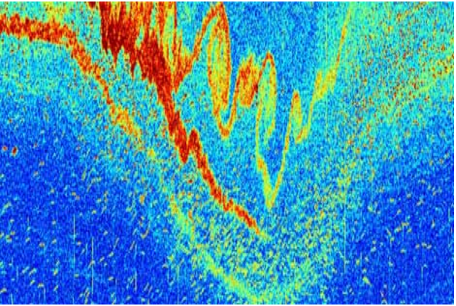


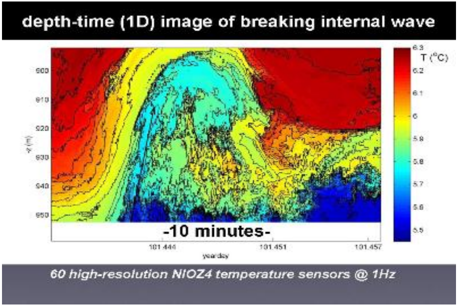


Play/Pause

Why do we care?

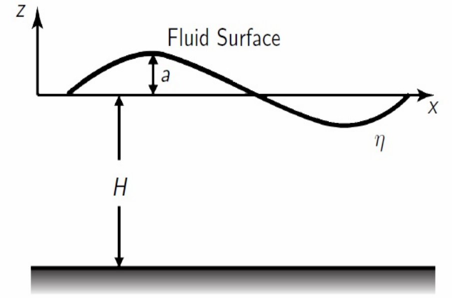
- Breaking internal waves affect the Meridional Overturning Circulation (thermohaline circulation)
- The spatial variability of mixing is imprtant for accurate climate modeling
- The Navy care about internal gravity waves; submarines don't want to get caught up in IGWs.





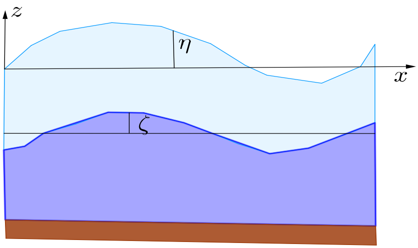
Hans van Haren

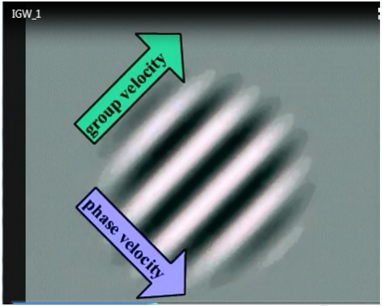
Lecture Plan



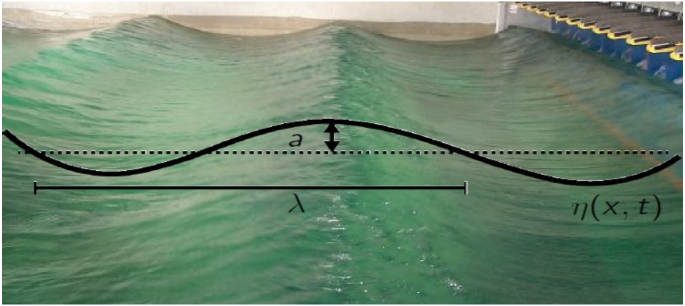
THEORY OF....

- Surface gravity waves
- Interfacial waves
- Internal waves in uniform stratification





Surface Gravity Waves: Mathematical Background

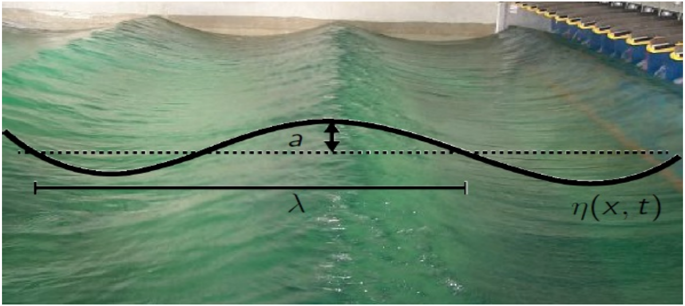


A simple way to describe a wave is

$$\eta(x, t) = a\cos(kx - \omega t)$$

- a is the amplitude
- k is the wavenumber ($k = 2\pi/\lambda$)
- ω is the frequency and $c = \omega/k$ is the phase speed

Surface Gravity Waves: Mathematical Background



For the wave $\eta(x,t) = a\cos(kx - \omega t)$

The phase speed is $c = \omega/k$

Important Fact

Waves of different wavenumbers may travel at different speeds.

Surface Gravity Waves: Solution Approach 1

A straightforward approach is to solve the full governing equations for an incompressible fluid:

$$\rho_0 \left(\frac{Du}{Dt} - f_0 v \right) = -\frac{\partial p}{\partial x}, \qquad p = \text{pressure}$$

$$\rho_0 \left(\frac{Dv}{Dt} + f_0 u \right) = -\frac{\partial p}{\partial y}, \qquad (u, v, w) = \text{velocity}$$

$$\rho_0 \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho, \qquad f_0 = \text{Coriolis frequency}$$

$$\rho_0 \frac{D\rho}{Dt} = -w \frac{d\bar{\rho}}{dz}, \qquad \rho(x, y, z, t) = \text{density}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad \bar{\rho}(z) = \text{background density}$$

We are not going to do that here :)

Surface Gravity Waves: Solution Approach 2

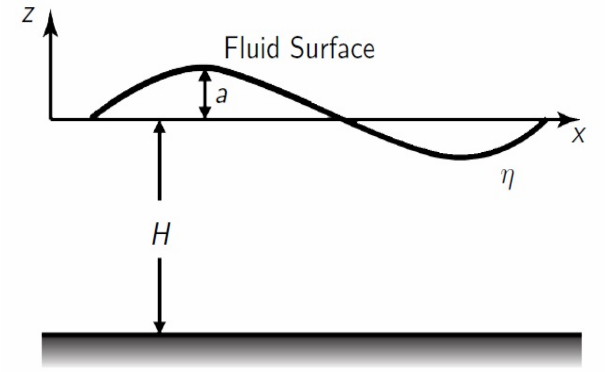
Rather than solve the previous equations, we first make simplifying assumptions about the flow field.

Assumptions

We consider

- Small amplitude waves
- Unaffected by Earth's rotation
- Inviscid - viscosity is negligible
- Incompressible - no sound waves allowed ($\nabla \cdot \mathbf{u} = 0$)
- Irrotational ($\nabla \times \mathbf{u} = 0$)
- Two dimensional: $\mathbf{u} = (u, w)$

Surface Gravity Waves: Setup



- η is the surface displacement of fluid from resting depth, $z = 0$
- a is the amplitude of η
- H is the resting depth of fluid

Surface Gravity Waves: Solution Approach 2

The fact that the fluid is irrotational allows us to define a so-called velocity potential, ϕ, such that

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad w = \frac{\partial \phi}{\partial z} \tag{1}$$

The mass conservation equation, $\nabla \cdot \mathbf{u} = 0$, then gives

Mass Equation (Laplace Equation)

$$\frac{\partial^2 \phi^2}{\partial x^2} + \frac{\partial^2 \phi^2}{\partial z^2} = 0 \tag{2}$$

Surface Gravity Waves: Solution Approach 2

To proceed further, we need to specify boundary conditons at the free surface and at the bottom. At the bottom, the normal (vertical) velocity is zero. Thus

Boundary condition at bottom

$$w = \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -H \tag{3}$$

Boundary condition at surface

$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \tag{4}$$

This is a linearized form of the so called **kinematic boundary condition**: fluid particles never leave the surface.

Surface Gravity Waves: Solution Approach 2

The time-dependence of ϕ comes from another surface boundary condition: the pressure at the free surface is equal to the ambient pressures (taken to be zero) such that

$$p = 0 \quad \text{at} \quad z = \eta$$

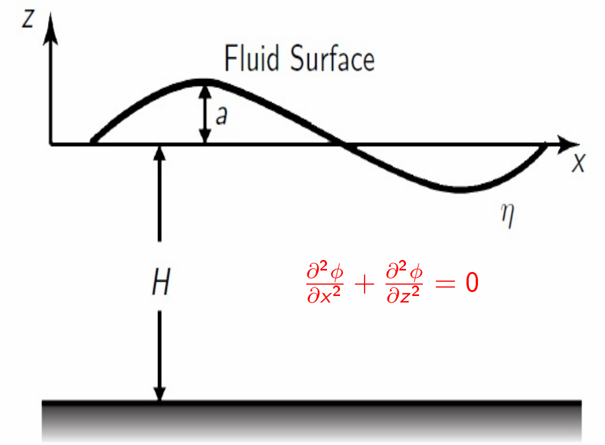
Applying a linearized form of Bernoulli's equation then gives:

Dynamic boundary condition at surface

$$\rho\frac{\partial\phi}{\partial t} + \rho g\eta = 0 \quad \text{at} \quad z = 0$$

$$\implies \frac{\partial\phi}{\partial t} = -g\eta \quad \text{at} \quad z = 0 \tag{5}$$

Surface Gravity Waves: Governing Equations



$$\frac{\partial \phi}{\partial t} = -g\eta \quad \text{at } z = 0$$
$$\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ at } z = -H$$

IMPORTANT

- The boundary conditions imply specifying a form for η.
- The kinematic condition imply separation of variables

Surface Gravity Waves: Solution

Let

$$\eta = a\cos(kx - \omega t) \tag{6}$$

and assume a separable solution of the form

$$\phi = \hat{\phi}(z)\sin(kx - \omega t) \tag{7}$$

Substituting (6) and (7) into the Laplace equation and using the kinematic and bottom boundary conditions yield, after some algebra,

$$\phi(x, z, t) = \frac{a\omega}{k}\frac{\cosh k(z + H)}{\sinh kH}\sin(kx - \omega t) \tag{8}$$

Surface Gravity Waves: Solution cont..

$$\phi(x, z, t) = \frac{a\omega}{k} \frac{\cosh k(z + H)}{\sinh kH} \sin(kx - \omega t) \quad (9)$$

Note that we haven't used the dynamic boundary condition yet. It gives an important relation between k and ω. Substituting our solution above and $\eta = a\cos(kx - \omega t)$ into the dynamic condition results in

Dispersion relation

$$\omega = \sqrt{gk \tanh kH} \quad (10)$$

Phase speed

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kH} \quad (11)$$

Approximations

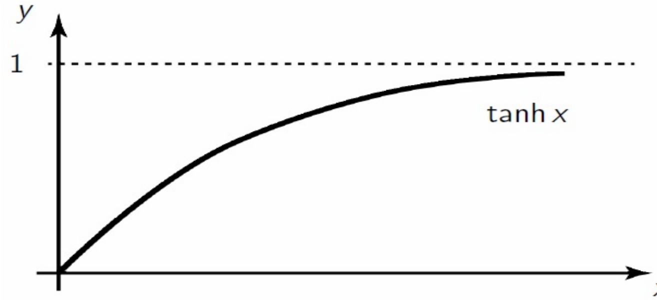
Using $\lambda = 2\pi/k$, we are interested in the cases when

- $H/\lambda \gg 1$ or $kH \gg 1$ (deep water)
- $H/\lambda \ll 1$ or $kH \ll 1$ (shallow water)

Note that

$$\tanh kH \to 1 \quad \text{when} \quad kH \gg 1$$

$$\tanh kH \to kH \quad \text{when} \quad kH \ll 1$$



Deep Water Approximation

Deep Water Waves

$$kH \gg 1 : \quad c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh kH} \to \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

c is dependent on k

- The wave does not feel the bottom.
- Longer waves in deep water propagate faster.
- Deep water waves are dispersive: A wave "packet" separates or disperses.

Shallow Water Approximation

Shallow Water Waves

$$kH \ll 1: \quad c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH} \to \sqrt{gH}$$

c is independent of k

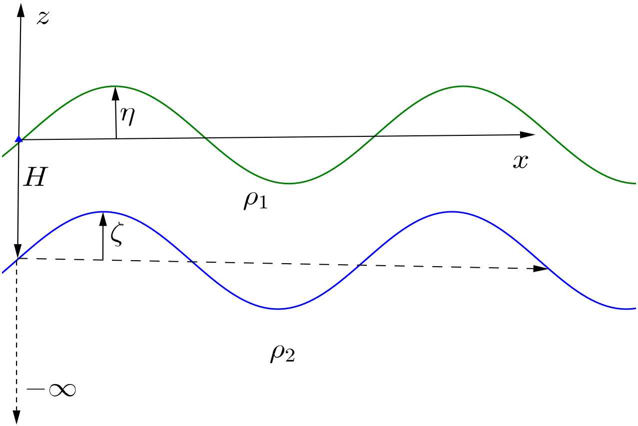
- The wave does feel the bottom; phase speed increases with water depth.
- Shallow water waves are not dispersive: A wave "packet" stays together.

1. Show that the solution:

$$\phi(x, z, t) = \frac{a\omega}{k} \frac{\cosh k(z + H)}{\sinh kH} \sin(kx - \omega t) \tag{12}$$

   satisfies the three boundary conditions.
2. Determine the u and w velocity components from (12).
3. What are the phase relationships between the velocities and the displacement, $\eta = a \cos(kx - \omega t)$?
4. Sketch one wavelength of η and illustrate (sketch) the phase relationships in question 3.

Interfacial Waves: Mathematical Background



Plan of Attack

- Solve the Laplace equation in both layers
- Apply the continuity of p and w at the interface

Interfacial Waves: Governing Equations

$$\frac{\partial^2 \phi_1}{\partial x} + \frac{\partial^2 \phi_1}{\partial z} = 0 \qquad (13)$$

$$\frac{\partial^2 \phi_2}{\partial x} + \frac{\partial^2 \phi_2}{\partial z} = 0 \qquad (14)$$

$$\phi_2 \to 0 \quad \text{at} \quad z \to -\infty \qquad (15)$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0 \qquad (16)$$

$$\frac{\partial \phi_1}{\partial t} = -g\eta \quad \text{at} \quad z = 0 \qquad (17)$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} = \frac{\partial \zeta}{\partial t} \quad \text{at} \quad z = -H \qquad (18)$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g\zeta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g\zeta \quad \text{at} \quad z = -H \qquad (19)$$

Interfacial Waves: Solution

Assume:

$$\eta = ae^{i(kx-\omega t)} \tag{20}$$

$$\zeta = be^{i(kx-\omega t)} \tag{21}$$

where a can be real but b should be left complex since η and ζ may not be in phase. The use of complex notation here is to simplify the algebra. Note that only the real part of the equation is meant.

Seek separable solutions

The boundary conditions imply separable solutions of the form

$$\phi_1 = \left(Ae^{kz} + Be^{-kz}\right) e^{i(kx-\omega t)} \tag{22}$$

$$\phi_2 = Ce^{kz} e^{i(kx-\omega t)} \tag{23}$$

Interfacial Waves: Solution

To solve for A, B and C, substitute (22)-(23) into the Laplace equations and use conditions (16)-(18) to get, after some algebra,

$$A = -\frac{ia}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) \tag{24}$$

$$B = \frac{ia}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right) \tag{25}$$

$$C = -\frac{ia}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) - \frac{ia}{2}\left(\frac{\omega}{k} - \frac{g}{\omega}\right)e^{2kH} \tag{26}$$

$$b = \frac{a}{2}\left(1 + \frac{gk}{\omega^2}\right)e^{-kH} + \frac{a}{2}\left(1 - \frac{gk}{\omega^2}\right)e^{kH} \tag{27}$$

Interfacial Waves: Solution

Solution

$$\phi_1 = -\frac{ia}{2}\left[\left(\frac{\omega}{k}+\frac{g}{\omega}\right)e^{kz} - \left(\frac{\omega}{k}-\frac{g}{\omega}\right)e^{-kz}\right]e^{i(kx-\omega t)}$$

$$\phi_2 = -\frac{ia}{2}\left[\left(\frac{\omega}{k}+\frac{g}{\omega}\right) + \left(\frac{\omega}{k}-\frac{g}{\omega}\right)e^{2kH}\right]e^{kz}e^{i(kx-\omega t)}$$

The velocities in the upper and lower layers can now be obtained via:

Velocities

$$(u_1, w_1) = \left(\frac{\partial \phi_1}{\partial x}, \frac{\partial \phi_1}{\partial z}\right)$$

$$(u_2, w_2) = \left(\frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_2}{\partial z}\right)$$

Interfacial Waves: Solution

Similar to the one-layer case, we now apply the dynamic boundary conditions, (19), to get the dispersion relation $\omega(k)$

Dispersion relation

$$\left(\frac{\omega^2}{gk} - 1\right)\left\{\frac{\omega^2}{gk}[\rho_1 \sinh kH + \rho_2 \cosh kH] - (\rho_2 - \rho_1)\sinh kH\right\} = 0$$

Unlike the one-layer case, here there are two possible roots given by the expressions

$$\left(\frac{\omega^2}{gk} - 1\right) \quad \text{and}$$

$$\left\{\frac{\omega^2}{gk}[\rho_1 \sinh kH + \rho_2 \cosh kH] - (\rho_2 - \rho_1)\sinh kH\right\}$$

Interfacial Waves: Barotropic Mode

The first expression gives the barotropic (surface) mode:

Barotropic Mode

$$\omega = \sqrt{gk}$$

$$\text{and (27) implies} \quad b = ae^{-kH}$$

Important:

- The barotropic mode behaves like deep water waves
- The amplitude at the interface is smaller than that at the surface by the factor e^{-kH}
- The motions of the free surface and interface are locked in phase. They move up and down simultaneously.
- The barotropic mode is similar to surface waves propagating in the upper layer fluid.

Interfacial Waves: Baroclinic Mode

The second expression gives the baroclinic (internal) mode:

Baroclinic Mode

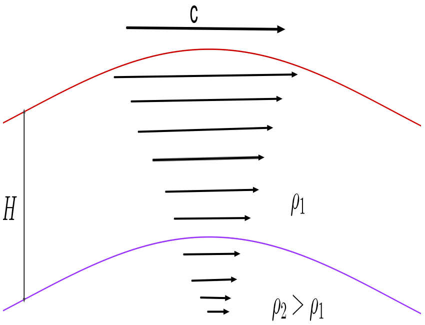
$$\omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh kH}{\rho_2 \cosh kH + \rho_1 \sinh kH} \tag{28}$$

$$\text{and (27) implies} \quad \eta = -\zeta \left(\frac{\rho_2 - \rho_1}{\rho_1} \right) e^{-kH} \tag{29}$$

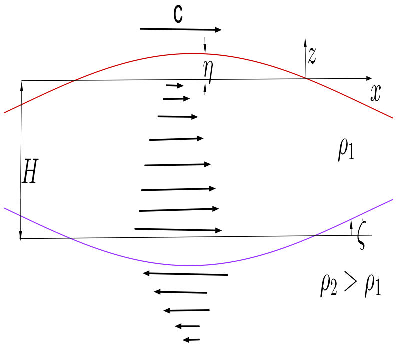
Important:

- η and ζ have opposite signs; implying that they are out of phase
- The interface displacement is much larger than the surface displacement if the density difference is small.
- It can be shown that the horizontal velocity changes sign across the interface.

Barotropic vs Baroclinic Mode



Barotropic mode



Baroclinic mode

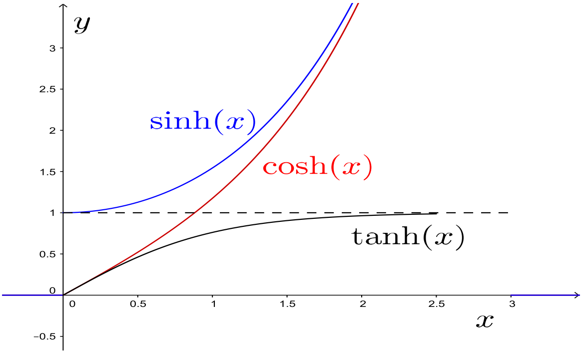
Shallow Water (Long Wave) Approximation

Assume wavelengths are large compared to the upper layer depth.

- $H/\lambda \ll 1$ or $kH \ll 1$ (shallow water)

$$\sinh kH \to kH \quad \text{when} \quad kH \ll 1$$

$$\cosh kH \to 1 \quad \text{when} \quad kH \ll 1$$



Shallow Water (Long Wave) Approximation

Then the dispersion relation (28) becomes

$$\omega^2 = \frac{k^2 Hg(\rho_2 - \rho_1)}{\rho_2} = k^2 Hg'$$

where g' is called the **reduced gravity** and defined as

$$g' = \frac{g(\rho_2 - \rho_1)}{\rho_2}$$

Phase Speed & Displacements

Thus, we get

$$c = \sqrt{g'H} \tag{30}$$

$$\eta = -\zeta\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) \tag{31}$$

Interfacial Waves: Exercises

(1) Compare and contrast the phase speed of interfacial waves to that of surface waves under the shallow water approximations.

(2) A **thermocline** is a thin layer in the upper ocean across the temperature, and consequently density, changes rapidly. Suppose the thermocline in a very deep ocean is at a depth of 100 m from the ocean surface, and that the reduced gravity is $0.025\,\mathrm{m/s^2}$.

a) Neglecting Coriolis effects, show that the speed of propagation of long gravity waves on such a thermocline is 1.58 m/s.

b) If the sursace displacement at a particular location is 2.3 m, determine the interface displacement at the same location.