Tides

Brian K. Arbic

Outline

1. Equilibrium tide, lunar vs solar vs sidereal days, spring-neap cycle, declination and diurnal tides
2. Harmonic decomposition
3. Tidal species
4. Laplace's tidal equations
5. Tidal response to astronomical forcing
6. Solid earth tides and effects of self-gravitation
7. Coastal tides

Reading: Stewart Ch. 17.4, 17.5

Motivation

• Although the study of tides dates back to Newton, they remain an important part of physical oceanography.

• Satellite altimetry has revolutionized our understanding of tides by producing amongst other things accurate global maps of tidal elevations.

• Tides are an important signal in current meter data, tide gauge records of sea level, and other instruments.

• They are important for navigation.

• Tides are an important source of mixing in both the coastal and open-ocean.

Equilibrium tide I: Cause

• Tidal forces are due to the differential force of gravity across a body of finite size–for instance the Earth rotating around the center of mass of the Earth-Moon system.

• The next five slides give a cartoon view, from a lower-level oceanography course (Essentials of Oceanography, Trujillo and Thurman)

Earth-Moon system rotation

- Barycenter between Moon and Earth revolves around Sun

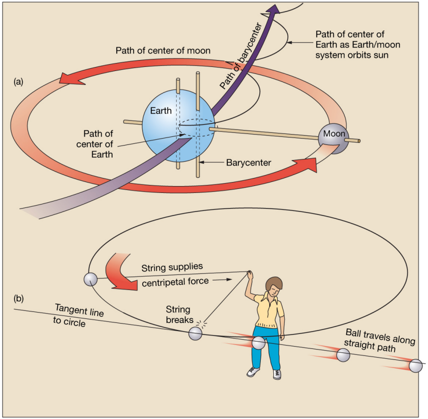


Fig. 9.1, page 278

Gravitational forces

- The gravitational force is largest on the side of the Earth closer to the moon and least on the side further from the Moon. The force is always directed toward the center of the Moon.

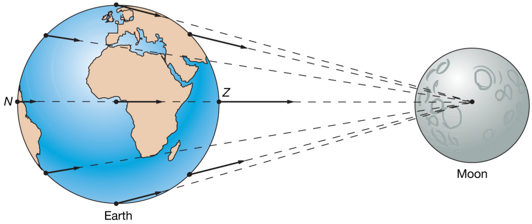


Fig. 9.2, p. 279

Centripetal force

- Due to rotation about "barycenter" or center of mass between two bodies in orbital motion.
- The centripetal force (the red arrows) is everywhere the same. The red arrows are all the same length and point in in the same direction.

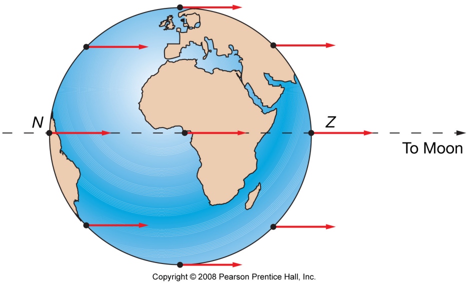


Fig. 9.3, p. 279

Tide-producing (resultant) forces

- Resultant forces = differences between centripetal and gravitational forces
- The **tidal generating force** (blue) is the **difference** of the **Moon's gravitational force** (black) and the **centripetal force** (red). Tide-generating forces point towards the Moon at Z (zenith) and away from the Moon at N (nadir).

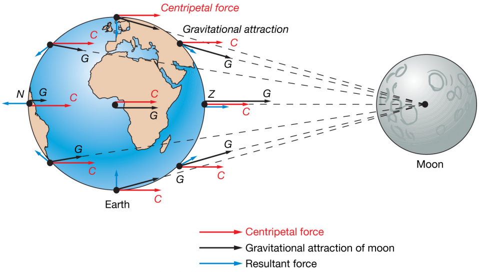


Fig. 9.4, p. 280

Equilibrium tide, Trujillo and Thurman

Tidal bulges (lunar): Ideal Earth covered by ocean

- One bulge faces Moon
- Other bulge on opposite side
- Tidal cycle results from earth rotating underneath these bulges→ two high tides and two low tides during a day
- The simple shape in the figure is called the equilibrium tide.
- Actual tide in oceans is a complex response to forcing of equilibrium tide

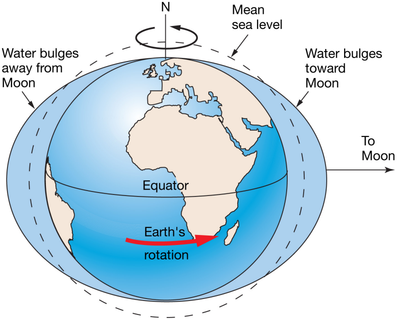


Fig. 9.6, p. 281

Equilibrium tide II: Mathematics

• All points on Earth experience the same centripetal force per unit mass due to motion around the center of mass of the Earth-Moon system:

$$F_{centripetal} = \frac{GM_{moon}}{r^2},$$

where G is Newton's gravitational constant, M_{moon} is the mass of the moon, and r is the distance from the center of the moon to the center of the Earth.

• On side of Earth closest to Moon, the gravitational pull of the Moon is given by

$$F_{gravitational} = \frac{GM_{moon}}{(r-a)^2},$$

where a is the radius of the Earth.

Equilibrium tide mathematics continued

$$Tidal\ force = F_{gravitational} - F_{centripetal} =$$

$$GM_{moon}[\frac{1}{(r-a)^2} - \frac{1}{r^2}] \approx$$

$$\frac{GM_{moon}}{r^2}[\frac{1}{1-\frac{2a}{r}} - 1] \approx \frac{2aGM_{moon}}{r^3}$$

- On side of Earth farthest Moon, we have

$$Tidal\ force = F_{gravitational} - F_{centripetal} =$$

$$GM_{moon}[\frac{1}{(r+a)^2} - \frac{1}{r^2}] \approx \frac{GM_{moon}}{r^2}[\frac{1}{1+\frac{2a}{r}} - 1] \approx \frac{-2aGM_{moon}}{r^3}$$

- The force is equal but opposite, yielding two bulges in the equilibrium tide.

Equilibrium tide III: Solar vs Lunar Tides

- The same reasoning applies to the Sun's gravity field on the earth.
- Therefore the sun should also cause tides.
- Show of hands: which are larger?
  - solar tides
  - lunar tides

Solar vs Lunar tides continued

$M_{sun} = 27{,}000{,}000 \ M_{moon}$
$r_{sun} = 390 \ r_{moon}$
$\frac{M_{sun}}{r_{sun}^3} = 0.45 \frac{M_{moon}}{r_{moon}^3}$

• Sun loses to Moon when considering differential gravity (cause of tides).

Period of principal solar vs principal lunar tides

- Is the period of the principal lunar tide equal to that of the principal solar tide?

- Show of hands
  - equal
  - not equal

Lunar day versus solar day (Trujillo and Thurman)

Key question—how is a lunar day different from a solar day? (p. 281)

- Moon orbits Earth
- While the Earth rotates around its axis, the Moon moves relative to the Earth, so
- 24 hours 50 minutes for observer to see subsequent Moons directly overhead (lunar day)
- High lunar tides are 12 hours and 25 minutes apart → period of principal lunar semidiurnal tide is 12 h 25 m
- 24 hours for observer to see subsequent Suns directly overhead (solar day)
- High solar tides are 12 hours apart → period of principal solar semidiurnal tide is 12 h

Lunar day versus solar day (Trujillo and Thurman)

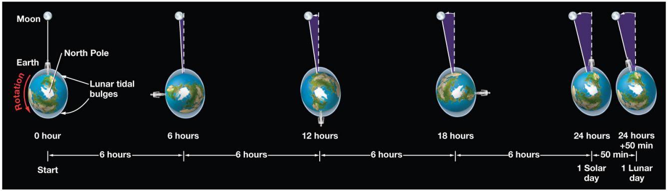


Fig. 9.7, p. 281

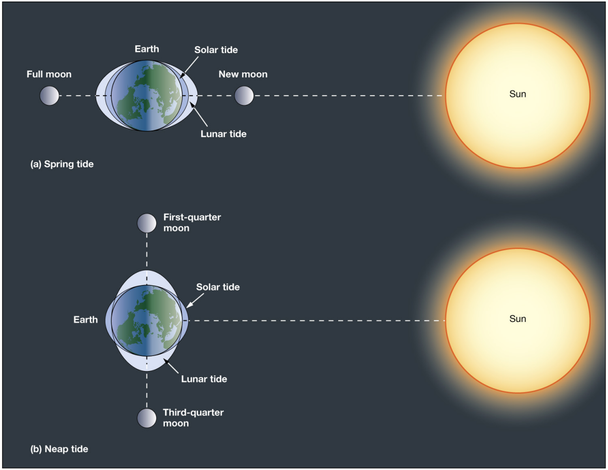
Equilibrium tide IV: Spring tide (Trujillo and Thurman)

Spring tides occur when the Moon, Earth and Sun are aligned (in syzygy, upper diagram on next page). Either new moon or full moon. Tidal range (difference between high and low tides) is large: exceptionally high high tides, low low tides.

Neap tides occur when Earth-Moon line is at right angles to Earth-Sun line (lower diagram on next page). Sun's bulges do not reinforce the Moon's bulges then. Tidal range is lower; high tides not so high, and low tides not so low. Occurs during first-quarter or third-quarter moon.

Spring tides occur twice a (lunar) month (not during Spring Season!!). Similarly, the neap tides occur twice a (lunar) month; twice each 29.5 days.

Spring tides continued (Trujillo and Thurman)



Equilibrium tide V: Diurnal tides

- Diurnal tides are due to the declination of the Moon's orbit to the equator.

- Declination yields asymmetries in the two high tides that occur every day.

- Mathematically, this is like having a forcing at once per day.

Declination (Trujillo and Thurman)

Declination

- Angular distance of Moon or Sun above or below Earth's equator
- Sun to Earth: 23.5º N or S of equator
- Moon to Earth: 28.5º N or S of equator

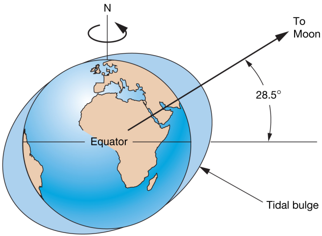


Fig. 9.11, page 284

- Shifts lunar and solar bulges from equator
- Diurnal inequality: Two high tides a day are of unequal size
→ Tidal forcing exists at periods near 24 h

Declination (Trujillo and Thurman)

Declination

- Angular distance of Moon or Sun above or below Earth's equator
- Sun to Earth: 23.5º N or S of equator
- Moon to Earth: 28.5º N or S of equator

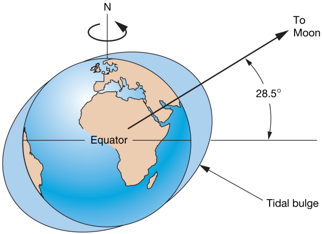


Fig. 9.11, page 284

- Shifts lunar and solar bulges from equator
- Diurnal inequality: Two high tides a day are of unequal size

→ Tidal forcing exists at periods near 24 h

Harmonic decomposition

- Frequencies of celestial motions:
  - $\omega_0 = \frac{2\pi}{1 \text{ mean solar day}}$
  - $\omega_1 = \frac{2\pi}{1 \text{ mean lunar day}} = \frac{2\pi}{1.0351 \text{ mean solar days}}$
  - $\omega_2 = \frac{2\pi}{1 \text{ sidereal month}} = \frac{2\pi}{27.3217 \text{ mean solar days}}$
  - $\omega_3 = \frac{2\pi}{1 \text{ tropical year}} = \frac{2\pi}{365.2422 \text{ mean solar days}}$
  - And others...
- Frequencies of some tidal constituents:
  - M_2: $2\omega_1$
  - S_2: $2\omega_0$
  - K_1: $\omega_{sidereal} = \omega_0 + \omega_3 = \omega_1 + \omega_2$
  - M_f: $2\omega_2$
  - And many others...
- Spring-neap cycle: frequency beating, especially M_2 and S_2

Tidal species

- The latitudinal and longitudinal dependence of the equilibrium tide η_{EQ} depends on the "tidal species" involved.
  - Semidiurnal tides (M_2,S_2,N_2,K_2):
    - $\eta_{EQ} = A(1 + k_2 - h_2)cos^2(\phi)cos(\omega t + 2\lambda)$
  - Diurnal tides (K_1,O_1,P_1,Q_1):
    - $\eta_{EQ} = A(1 + k_2 - h_2)sin(2\phi)cos(\omega t + \lambda)$
  - Long-period tides (M_f, M_m):
    - $\eta_{EQ} = A(1 + k_2 - h_2)[\frac{1}{2} - \frac{3}{2}sin^2(\phi)]cos(\omega t)$
- λ is longitude with respect to Greenwich
- ϕ is latitude
- t is time
- A and ω are constituent-dependent amplitudes and frequencies, respectively.
- The factor $1+k_2$-h_2 will be discussed later.

Ten commonly considered constituents

Const.	ω (10^{-4} s^{-1})	A (cm)	Period (solar days)
M_m	0.026392	2.2191	27.5546
M_f	0.053234	4.2041	13.6608
Q_1	0.6495854	1.9273	1.1195
O_1	0.6759774	10.0661	1.0758
P_1	0.7252295	4.6848	1.0027
K_1	0.7292117	14.1565	0.9973
N_2	1.378797	4.6397	0.5274
M_2	1.405189	24.2334	0.5175
S_2	1.454441	11.2743	0.5000
K_2	1.458423	3.0684	0.4986

Question: Can the actual ocean tide equal the equilibrium tide?

- Or, asking it in a different way: are there any factors you can think of that might prevent the actual ocean tides from attaining the equilibrium state?

Reasons for actual tides to be different from equilibrium tides

- Friction
- Earth's rotation
- Continents in the way
- Wave speed not fast enough (homework)

• The response is complex and varies from place to place.

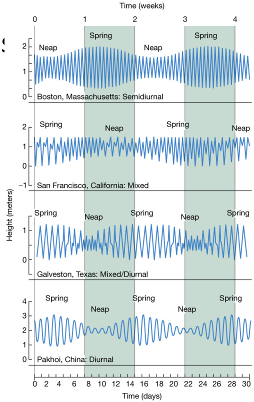
• In fact, in some locations, the dominant tide is diurnal (next slide).

Tidal types (Trujillo and Thurman)

Monthly tidal curves

- During a month there are two spring tides and two neap tides at all locations.
- Boston has semi-diurnal tides
- San Francisco and Galveston have diurnal/mixed tides
- Pakhoi, China has diurnal tides

Fig. 9.16, p. 290



Quantitative modeling of tidal response

• We model the tidal response to astronomical forcing with the shallow-water equations forced by the equilibrium tide.

• Such equations are traditionally known as the Laplace Tidal Equations.

Laplace's tidal equations

- Laplace: actual tide is dynamical response to equilibrium forcing.
- Modern shallow-water models often include nonlinearities and friction:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u} + f\hat{k} \times \vec{u} = -g\nabla(\eta - \eta_{EQ}) + \mathit{Friction}$$

$$\frac{\partial \eta}{\partial t} + \nabla \bullet [(H + \eta)\vec{u}] = 0$$

- In recent forward tide models, *Friction* includes eddy viscosity, quadratic bottom drag $\frac{c_d|\vec{u}|\vec{u}}{H}$, and a term having to do with energy conversion into internal waves over rough topography.
- Note that the equilibrium tide is the reference which the sea surface elevations η are measured against. The equilibrium tide is a perturbation to the geoid (equipotential).

Tidal response

• For each constituent:
$Tidal\, elevation(\phi, \lambda) = Amplitude(\phi, \lambda) cos[\omega t - phase(\phi, \lambda)]$,
phase wrt Greenwich

• We can construct maps of the amplitude and phase of tidal elevations, for each constituent.

• Note spring tidal range is the *peak-to-peak* value, which is twice the sum of the amplitudes of several constituents.

Kelvin Waves

• Kelvin waves are gravity waves in the presence of rotation and land/ocean boundaries.

• The theory of Kelvin waves is omitted here for the sake of brevity.

• Kelvin waves decay away from boundaries (land) with an e-folding decay scale of $\frac{\sqrt{gH}}{f}$, where $g = 9.8ms^{-2}$ is gravitational acceleration, H is water depth, and the Coriolis parameter $f = 2\Omega * sin(latitude)$, where $\Omega = \frac{2\pi}{day}$.

• Kelvin waves rotate counterclockwise in the Northern Hemisphere, clockwise in the Southern Hemisphere.

Kelvin waves and the direction of tidal propagation

- Type "TPXO" in google to obtain

http://volkov.oce.orst.edu/tides

- Class exercise:

1) are the largest tidal amplitudes generally located along coastlines or in the deep ocean? Is this consistent with Kelvin wave theory?
2) are the horizontal scales of tides in coastal areas versus the open ocean consistent with the theory?
3) does the Kelvin wave theory correctly describe the sense of rotation for:

  - African and European coasts of North Atlantic
  - Hudson Bay
  - North American coast of North Pacific
  - Southern Ocean (around Antarctica)
  - Argentine Atlantic coast (Patagonia)
  - New Zealand (this is an interesting case–is it clockwise?)

Solid-earth body tides

• Earth elastically yields to astronomical forcing. Since waves in solid earth are very fast, response is equal to equilibrium tide, times Love number h_2 i.e. $\eta_{bodytide} = h_2 \eta_{EQ}$

• Mass redistribution perturbs gravitational potential by amount equal to equilibrium tide times a different Love number k_2.

• How large are these effects?

Solid-earth body tides continued

• Solid-earth body tides can be as large as 10 cm.

• Hendershott (1972) adjusted numerical ocean tide models for this effect: η_{EQ} must be adjusted by factor $1 + k_2 - h_2 \sim 0.7$. These are called "Love numbers" after the scientist who first described them.

• Measured relative to moving seafloor, ocean tides are about 30% smaller than they would be if earth were perfectly rigid.

Love numbers for ten commonly considered constituents

Const.	ω (10^{-4} s^{-1})	A (cm)	Period (solar days)	1+k_2-h_2
M_m	0.026392	2.2191	27.5546	0.693
M_f	0.053234	4.2041	13.6608	0.693
Q_1	0.6495854	1.9273	1.1195	0.695
O_1	0.6759774	10.0661	1.0758	0.695
P_1	0.7252295	4.6848	1.0027	0.706
K_1	0.7292117	14.1565	0.9973	0.736
N_2	1.378797	4.6397	0.5274	0.693
M_2	1.405189	24.2334	0.5175	0.693
S_2	1.454441	11.2743	0.5000	0.693
K_2	1.458423	3.0684	0.4986	0.693

- Love numbers differ for diurnal tides, especially K_1, due to "free-core nutation resonance" (Wahr 1981; Wahr and Sasao 1981)

Self-attraction and loading

• Building on work of Lamb, Munk and MacDonald, Hendershott (1972) also considered yielding of earth to loading by ocean tides, changes in gravitational potential due to this yielding, and self-attraction of ocean tide upon itself, cumulatively known as "self-attraction and loading" term.

• Computed based on spherical harmonics, the natural basis functions on a sphere.

- Momentum equation with self-attraction and loading included is

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \bullet \nabla \vec{u} + f\hat{k} \times \vec{u} = -g\nabla(\eta - \eta_{EQ} - \eta_{SAL}) + \\ Friction,$$

where η_{EQ} has been modified by factor of $1 + k_2 - h_2$ and $\eta_{SAL} = \sum_n (1 + k_n' - h_n') \frac{3\rho_{water}}{\rho_{earth}(2n+1)} \eta_n$ (h_n' and k_n' are "load numbers", η_n is spherical harmonic decomposition of η).

- Important and expensive correction for ocean tide models.

Coastal tides

- The largest tidal elevations (height changes), and the largest tidal currents, are found in coastal areas.

- Typical tidal current values:
  - Up to $\sim$ 1 m s^{-1} in coastal areas
  - Typically about 2 cm s^{-1} in the open ocean

- Typical values of constituent elevation amplitudes:
  - Up to $\sim$ 4 m in coastal areas
  - Typically about 0.2-1 m in the open ocean

Large coastal tides (Trujillo and Thurman)

• Spring tidal range can be up to 17 m in regions of large tides such as the Bay of Fundy and the Hudson Strait.

• Note spring tidal range is the *peak-to-peak* value, which is twice the sum of the amplitudes of several constituents.